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Reduced Order Observer Based Pole Placement Design for Inverted Pendulum on Cart

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Abstract

The inverted pendulum is an under actuated system and unstable system without a controller. In this paper modelling of an inverted pendulum is done using the Euler-Lagrange equation for stabilization of it. The controller gain is evaluated through state feedback and reduced-order observer design techniques and the result for the different initial conditions is compared. The state feedback controller is designed by Pole- placement technique for different desired pole locations. The simulation of the inverted pendulum based on reduced order pole placement design has been done on MATLAB/SIMULINK. It has been observed from the simulation result that the angular velocity and cart speed tracks the system response for different initial conditions by varying the desired pole location for the left-hand plane of the s-plane. In general, if some of the systems are unknown and the other state is known we can design using a reduced-order observer for any physical system.

Keywords

Inverted pendulum, Pole placement, State feedback, Reduced order, Euler-Lagrange

1. Introduction

The concept of inverted pendulum is become important in everyday life especially in the field of control system engineering [1]. The inverted pendulum is the typical control system problem. The concept of stability that shows in inverted pendulum is often applied in real-world applications for instance the helicopter already uses concept stability to reject windup disturbance and also the missile that moving faster without having problem thanks to the concept of stability. It has some concepts like a hand as a cart and sticks as a pendulum which a hand is trying to balance the stick. In addition, the inverted pendulum has limited motion that only can move right and left meanwhile the hand which attempts to balance the stick has the advantage can moving upward and downward. An inverted pendulum does basically the same thing just like the broomstick, an inverted pendulum is an inherently unstable system [1, 2, and 3].

In [4-9] simulation of inverted pendulum has been tested using full sate feedback and the gain of the state feedback has been obtained using linear quadratic regulator and pole placement control. Full state estimator for inverted pendulum on the cart has been simulated and designed. The gain of the state feedback gain K has been obtained using linear quadratic regulator and the gain of full state observer L has been obtained using pole placement [11].

State feedback control of inverted pendulum has been simulated and due to nonlinearity of the system the authors linearized using Taylor series and Type-I servo system-based sate feedback technique has been tested and analyzed on MATLAB SOFTWARE in [12]. But they assumed that all state variables were estimated. In [12] they didn't consider if some of the state variables are not estimated.

In [7,10, 13-15] Linear quadratic regulator and pole placement control for stabilizing a cart inverted pendulum system has been simulated and compared on MATLAB Software. Both control mechanisms produce optimal system response with fast response and optimal control signal has been tested in [13]. This paper discusses modelling and simulation of reduced order-based pole placement for inverted pendulum. The equations of inverted pendulum are linearized by using Lagrange equation and the simulation results are tested for different initial conditions.

2. Mathematical Modelling of Inverted pendulum

The inverted pendulum is a typical problem in dynamics and control engineering, and it is commonly used as a standard for testing control algorithms on it [1]. Let the new coordinate of the center of gravity of the pendulum be X_p or $(x - l\sin\theta, l\cos\theta)$. Let us define the angle of the rod from the vertical (reference) line as θ and displacement of the cart as x. Also assume F the force applied to the system, g be the acceleration due to gravity and l be the length of the pendulum rod, v and ω be the translational and angular velocity of the cart and pendulum [2].

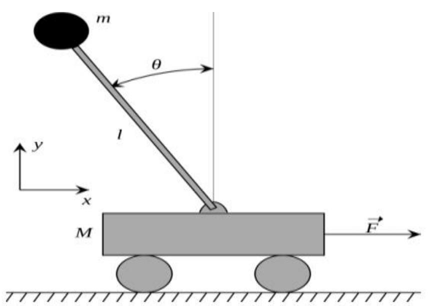

Figure 1. Inverted pendulum system

$$X_{c,i} = x, X_{c,j} = 0,$$
$$X_{p,i} = x - lsin\theta,\ X_{p,j} = lcos\theta \tag{1}$$

$$X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X_c = \begin{bmatrix} x \\ 0 \end{bmatrix}, X_p = \begin{bmatrix} x - lsin\theta \\ lcos\theta \end{bmatrix} \tag{2}$$

Where X_0=Origin, X_c =Position Vector of cart, and X_p =Position vector of Pendulum

Potential Energy of system $\mathrm{P.E} = mgl + MgX_{c,j} = mglcos\theta$

Kinetic Energy of system $\mathrm{K.E} = \frac{1}{2}\mathrm{M}\,\dot{X}_c{}^2 + \frac{1}{2}\mathrm{m}\,\dot{X}_p{}^2 + \frac{1}{2}\mathrm{J}\dot{\theta}^2$

$$\dot{X}_p{}^2 = V_x{}^2 + V_y{}^2 \quad (3)$$

$$V_x = \dot{x}^2 - 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\cos^2\theta \quad (4)$$

$$V_y = l^2\dot{\theta}^2\sin^2\theta \quad (5)$$

$$\dot{X}_p{}^2 = \dot{x}^2 - 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta$$

$$K.E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta) + \frac{1}{2}J\dot{\theta}^2$$

$$K.E = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}\dot{\theta}l\cos\theta + \frac{1}{2}(J + ml^2)\dot{\theta}^2 \quad (6)$$

$$P.E = mgl\cos\theta \quad (7)$$

The Euler Lagrange equation of motion for mechanical systems is given by

$$L = K.E - P.E$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}\dot{\theta}l\cos\theta + \frac{1}{2}(J + ml^2)\dot{\theta}^2 + mgl\cos\theta \quad (8)$$

The Euler Lagrange equation are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Resultant\ force\ (torque)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F - bx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Simplifying the above equation, we get

$$(M+m)\ddot{x} - m\ddot{\theta}l\cos\theta + ml\dot{\theta}^2\sin\theta + b\dot{x} = F \quad (9)$$

$$(J + ml^2)\ddot{\theta} - m\ddot{x}l\cos\theta - mgl\sin\theta = 0 \quad (10)$$

The above equation shows the dynamics of the entire system. In order to derive the linear differential equation modelling, we need to linearize the nonlinear differential equation obtained as above so far. For small angle deviation around the upright equilibrium point assume $\sin\theta = \theta, \cos\theta = 1, \dot{\theta}^2 = 0$

The simplified equations are shown below

$$(M+m)\ddot{x} - ml\ddot{\theta} + b\dot{x} = F \quad (11)$$

$$(J + ml^2)\ddot{\theta} - ml\ddot{x} - mgl\theta = 0 \quad (12)$$

Let us assign $M + m = p, ml = d, J + ml^2 = a, mgl = f, k = ap - d^2$

Let the state variable $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$ and the output of the system are the position of the cart and the angular position of the pendulum.

The state space representation of inverted pendulum as shown below

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \ddot{x} = -\frac{ab}{k}x2 + \frac{df}{k}x3 + \frac{a}{k}u$$

$$\dot{x_3} = x_4$$

$$\dot{x_4} = \ddot{\theta} = -\frac{db}{k}x2 + \frac{pf}{k}x3 + \frac{d}{k}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -\frac{ab}{k} & \frac{df}{k} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{db}{k} & \frac{pf}{k} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ \frac{a}{k}\\ 0\\ \frac{d}{k}\end{bmatrix}u \tag{13}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}u \tag{14}$$

Table 1. The parameters of the system

Parameter	***Value(Unit)***
$\boldsymbol{Mass\ of\ cart(M)}$	$0.5Kg$
$\boldsymbol{Mass\ of\ the\ pendulum(m)}$	$0.2Kg$
$\boldsymbol{Length\ of\ the\ pendulum(l)}$	$0.3m$
$\boldsymbol{Displecement\ of\ cart(x)}$	-
$\boldsymbol{Pendulum\ angle(\theta)}$	-
$\boldsymbol{Pendulum\ moment\ of\ inertia(J)}$	$0.006Kgm^2$
$\boldsymbol{Acceleration\ due\ to\ gravity(g)}$	$9.8m/s^2$
$\boldsymbol{Cofficent\ of\ friction\ of\ the\ cart(b)}$	$0.1Ns/m$

2.1. Minimum order state observer

The state vector for inverted pendulum is an n -vector and the output vector y is an m -vector that can be measured. Since m output variables are linear combinations of the state variables, m state variables need not be estimated so we need to estimate only $n-m$ state variables. Then the reduced-order observer becomes a $(n-m)$th-order *observer* [3]. For this paper the state vectors are four (i.e $x, \dot{x}, \theta, \dot{\theta}$) and the output vector that can be measured are two (that is the linear position x and the angular position θ). Let the state variable x_a is equal to the output y those can be directly measured and state variable x_b is the unmeasurable portion of the state vector. Then the partitioned state and output equations become

$$\begin{bmatrix}\dot{x}_a\\ --\\ \dot{x}_b\end{bmatrix}=\begin{bmatrix}A_{aa} & A_{ab}\\ --- & ---\\ A_{ba} & A_{bb}\end{bmatrix}\begin{bmatrix}x_a\\ ---\\ x_b\end{bmatrix}+\begin{bmatrix}B_a\\ --\\ B_b\end{bmatrix}u$$

$$y=\begin{bmatrix}C_a & C_b\end{bmatrix}\begin{bmatrix}x_a\\ --\\ x_b\end{bmatrix} \tag{15}$$

Where, $x_a=\begin{bmatrix}x_1\\ x_3\end{bmatrix}$, $\quad x_b=\begin{bmatrix}x_2\\ x_4\end{bmatrix}$

Based on the above partitioned for measured state and unmeasured state the partitioned matrices as shown below.

$$A_{aa}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ A_{ab}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ A_{ba}=\begin{bmatrix}0 & \frac{df}{k}\\ 0 & \frac{pf}{k}\end{bmatrix},\ A_{bb}=\begin{bmatrix}\frac{-ab}{k} & 0\\ -\frac{db}{k} & 0\end{bmatrix},\ B_a=\begin{bmatrix}0\\ 0\end{bmatrix},\ B_b=\begin{bmatrix}\frac{a}{k}\\ \frac{d}{k}\end{bmatrix}$$

$C_a=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $C_b=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

The minimum order state equation

$$\dot{\tilde{\mu}} = (A_{bb} - K_e A_{ab})\tilde{\mu} + [(A_{bb} - K_e A_{ab})Ke + (A_{ba} - K_e A_{aa})]y + (B_b - K_e B_a)u \quad (16)$$

$$\dot{\tilde{\mu}} = (A_{bb} - K_e A_{ab})\tilde{\mu} + [(A_{bb} - K_e A_{ab})Ke + (A_{ba} - K_e A_{aa})]y + (B_b - K_e B_a)u$$

The general minimum order state equation $\dot{\tilde{\mu}} = \hat{A}\tilde{\mu} + \hat{B}y + \hat{F}u$

$$\hat{A} = (A_{bb} - K_e A_{ab})$$
$$\hat{B} = [(A_{bb} - K_e A_{ab})Ke + (A_{ba} - K_e A_{aa})]$$
$$\hat{F} = (B_b - K_e B_a)$$

The minimum order output equation is

$$\tilde{x}\begin{bmatrix} x_a \\ -- \\ \tilde{x_b} \end{bmatrix} = \begin{bmatrix} y \\ -- \\ \tilde{x_b} \end{bmatrix} = \begin{bmatrix} zeros(2,2) \\ ----- \\ eye(2) \end{bmatrix}\tilde{\mu} + \begin{bmatrix} eye(2) \\ --- \\ K_e \end{bmatrix} y \quad (17)$$

The general minimum order output equation (Transformation of $\tilde{\mu}$ to $\tilde{x}$ *)* is $\tilde{x} = \hat{C}\tilde{\mu} + \hat{D}y$

Where, $\hat{C} = \begin{bmatrix} zeros(2,2) \\ ----- \\ eye(2) \end{bmatrix}$, $\hat{D} = \begin{bmatrix} eye(2) \\ --- \\ K_e \end{bmatrix}$

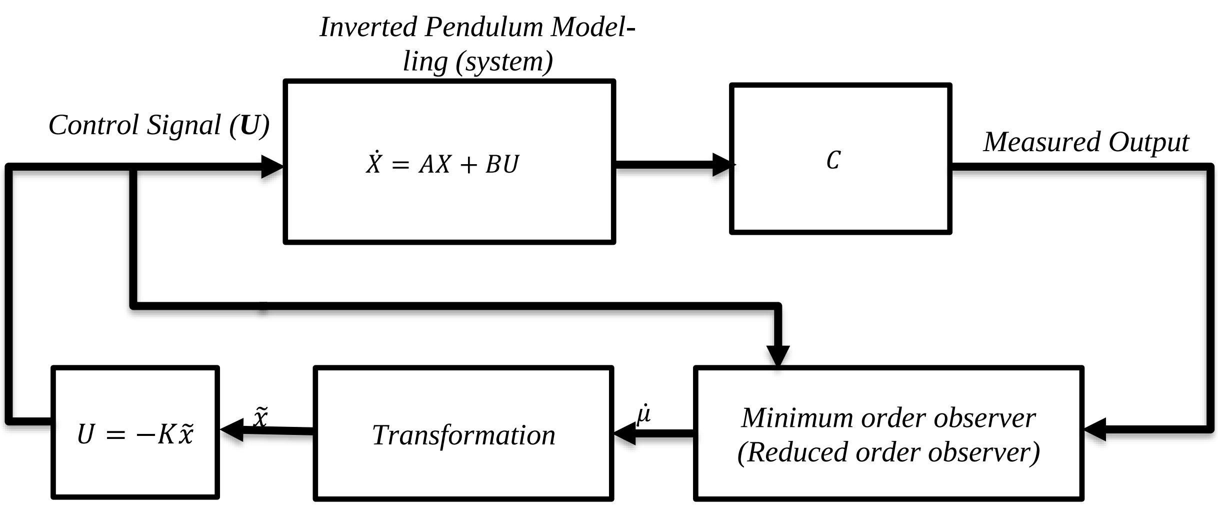

Figure 2. Block diagram for reduced order observer and feedback control design

3. Controller Design

In this paper two ways of controller were designed i.e pole placement for state feedback and observer gains.

3.1 Pole placement design

To design a pole placement is a method to employ to place the closed loop poles of a plant. Let us assume that the desired closed-loop poles are to be at $s = \mu_1, s = \mu_2, s = \mu_3, s = \mu_4 \ldots s = \mu_n$ [3]. The main goal of a pole placement design is to stabilize a system if it is initially unstable, or to improve the stability. By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable. The necessary and sufficient condition for pole placement design is the system must be completely controllable.

If the system is completely controllable if the rank of the controllable matric

$Qc = [B\ \ AB\ \ A^2B..\ \ \ A^{n-1}B$ has full rank n. For this paper the rank of the controllable matrix $Qc = 4$.

We can check using MATLAB command.

```
Qc=ctrb(A, B);
        if(rank(Qc)==4)
        disp('System is controllable')
          else
          disp('System is Uncontrollable')
          end
```

Therefore, the output of this paper is System is controllable

For this paper let the desired poles to be place at $s = -4 + j4, s = -4 - j4, s = -20, s = -20$

To design a full state feedback using pole placement by using Ackermann's or place command on MATLAB script that is $K = acker(A, B, d)$ or $K = place(A, B, d)$, where A, B are the system matrices and d is the desired poles of the system. The full state feedback controller gains after running the MATLAB code using the above desired poles are $K = -287.3469\ \ -100.6714\ \ \ 287.2388\ \ \ 50.7886$

The corresponding values are $K = [K1\ \ \ K2\ \ \ K3\ \ \ \ K4]$.

3.2 Reduced order observer design

The necessary and sufficient condition for pole placement design is the system must be completely controllable. Similarly for observability if the system is completely observable if the rank of the observable matrix

$Qo = [C\ \ CA\ \ CA^2..\ \ \ CA^{n-1}]^T$ has full rank n. For this paper the rank of the observable matrix $Qo = 4$.

The general MATLAB command to check the observability using Qo=obsv (A, C); But from the above reduced order equation the state matrix A equivalent to Abb and output matrix C equivalent to Aab. Therefore, to determine the reduced order observer design using Qor:

```
Qor=obsv(Abb, Aab);
        if(rank(Qor)==2)
        disp('The reduced order observer is observable')
        else
        disp('The reduced order observer is Unobservable')
        end
```

Therefore, the output of this paper is: The reduced order observer is observable. To design a reduced order observer system by assigning the unestimated states at a desired location to be $s = -5, s = -5$ using Ackermann's or place command on MATLAB script that is $Ke = acker(A^T, C^T, d)^T$ or $K = place(A^T, C^T, d)^T$, where A, B are the system matrices and d is the desired poles of the unestimated state. The reduced order gains Ke after running the MATLAB code using the above desired poles are

$$Ke = \begin{bmatrix} 4.8182 & 0 \\ -0.4545 & 5.0000 \end{bmatrix}$$

4. Simulation Result and Discussion

The general block diagram for this paper on MATLAB SIMULIUNK as shown in figure below

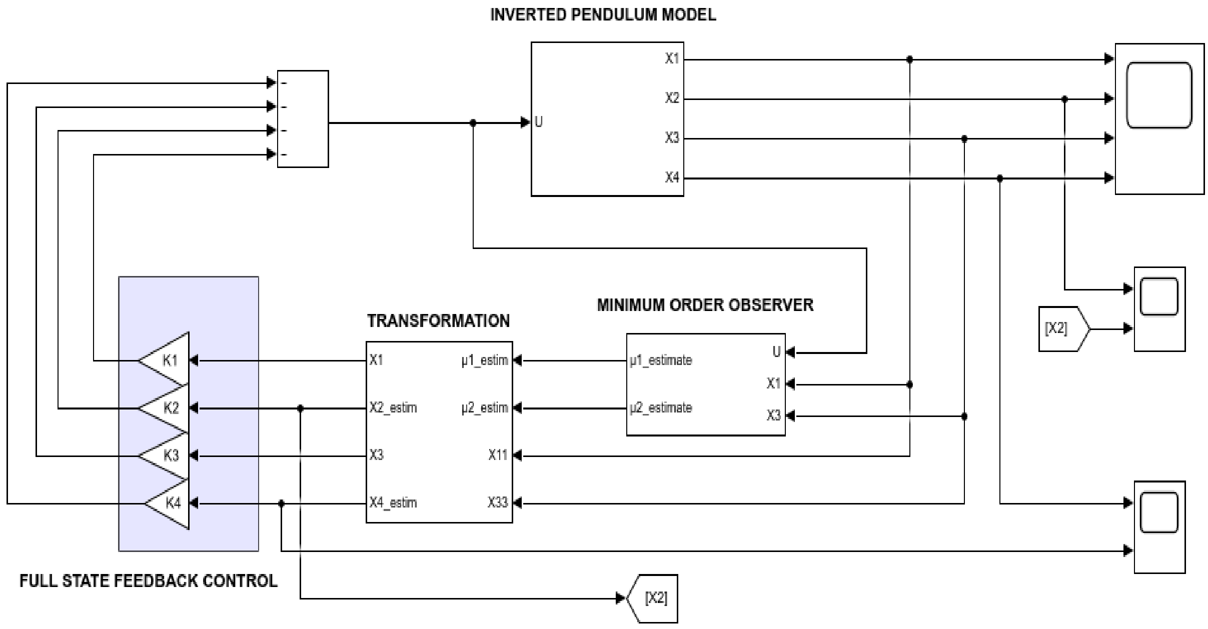

Figure 3. The general block diagram for reduced order observer design

The general block diagram for reduce order observer-based pole placement controller for inverted pendulum has shown in the above fig 3 and the transformation subsystem block drawn from equation (16) and the minimum order observer drawn from equation (17) has been simulated on the MATLAB Simulink.

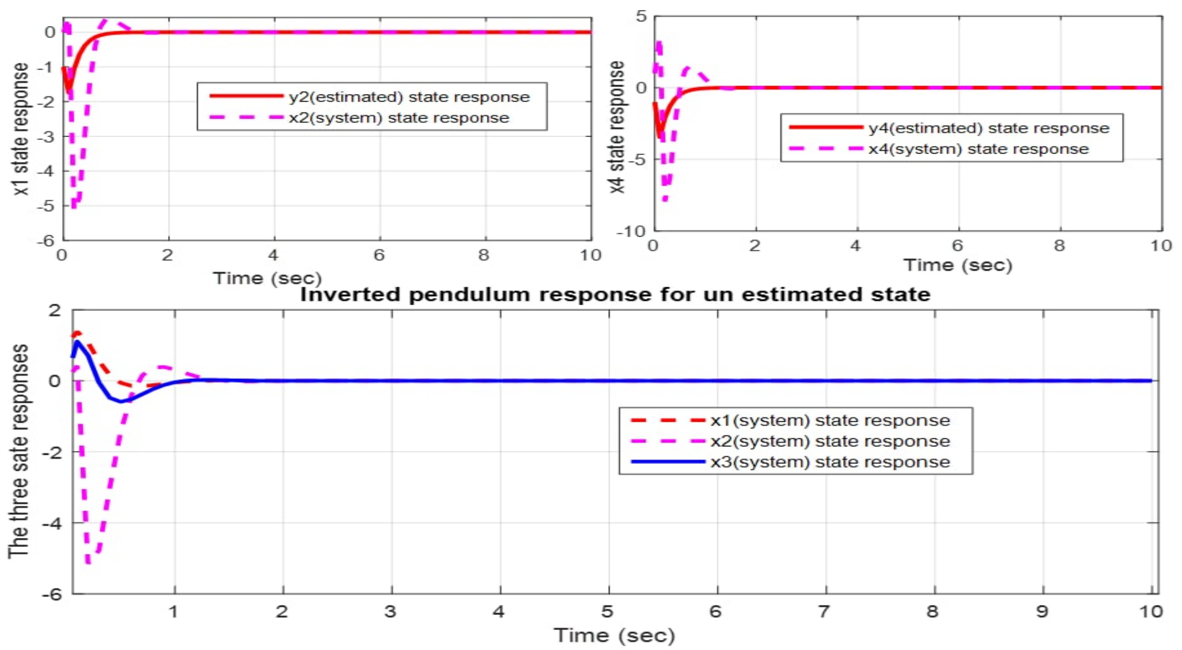

Figure 4. The response of the estimated and system response for initial conditions (1, 0, 0, 1)

As shown on fig 4 the simulation result for estimated states has been simulated and compared on MATLAB software. The estimated state behaves like the actual state. This is due to the effectiveness of the controller to estimate the unknown state has been simulated and compared.

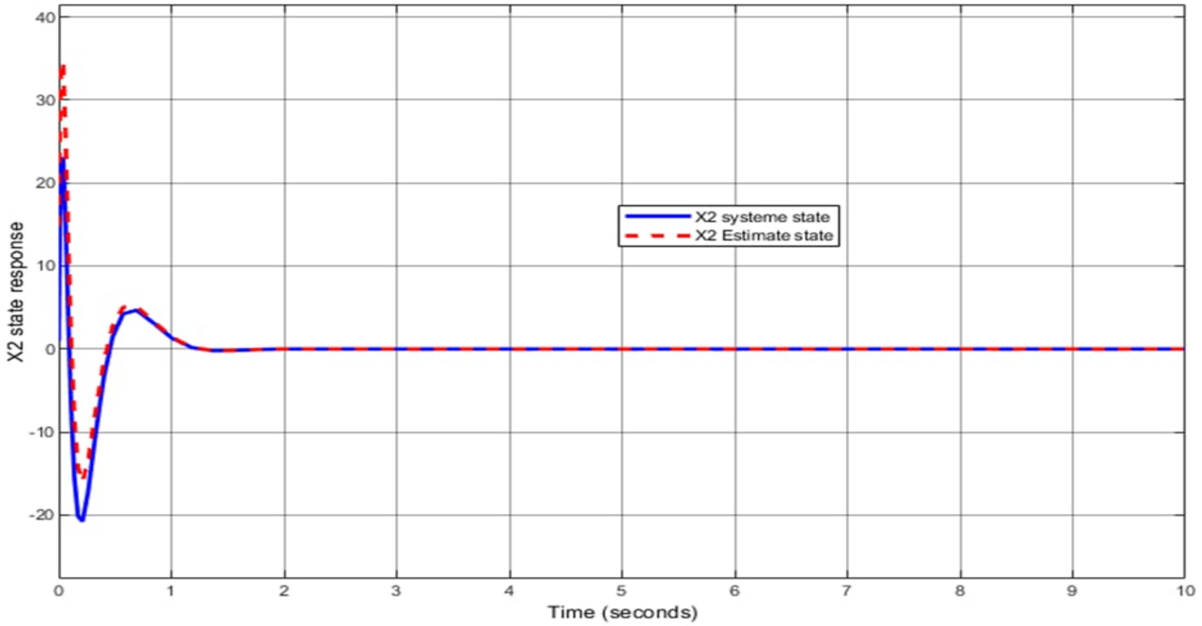

Figure 5. The response of the estimated and system response for x2 state

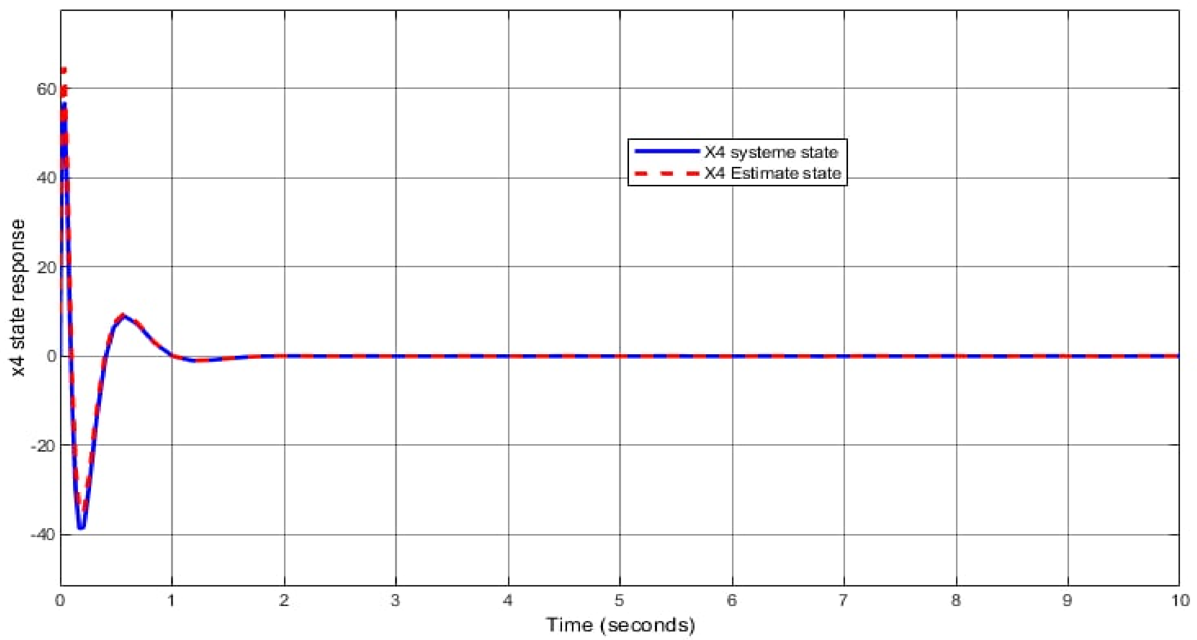

Figure 6. The response of the estimated and system response for x4 state

The simulation result on figure 5 and Figures 6 are captured from MATLAB Simulink and the simulation result is the same with that of simulated from MATLAB script on figure 4.

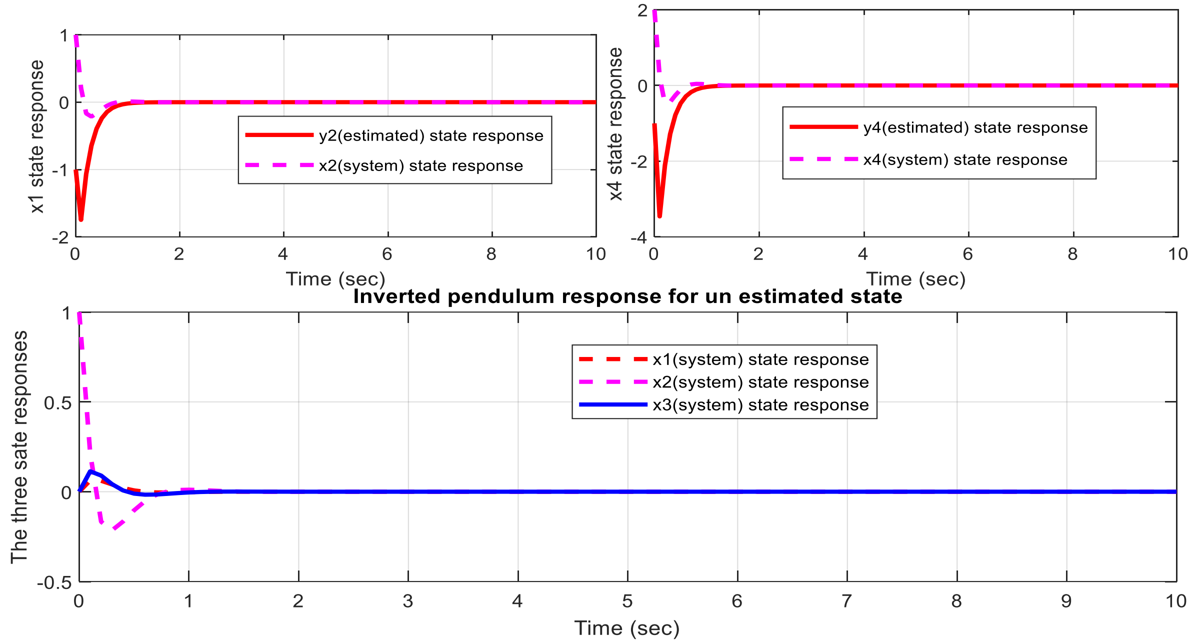

Figure 7. The response of the estimated and system response for initial conditions (0, 1, 0, 2)

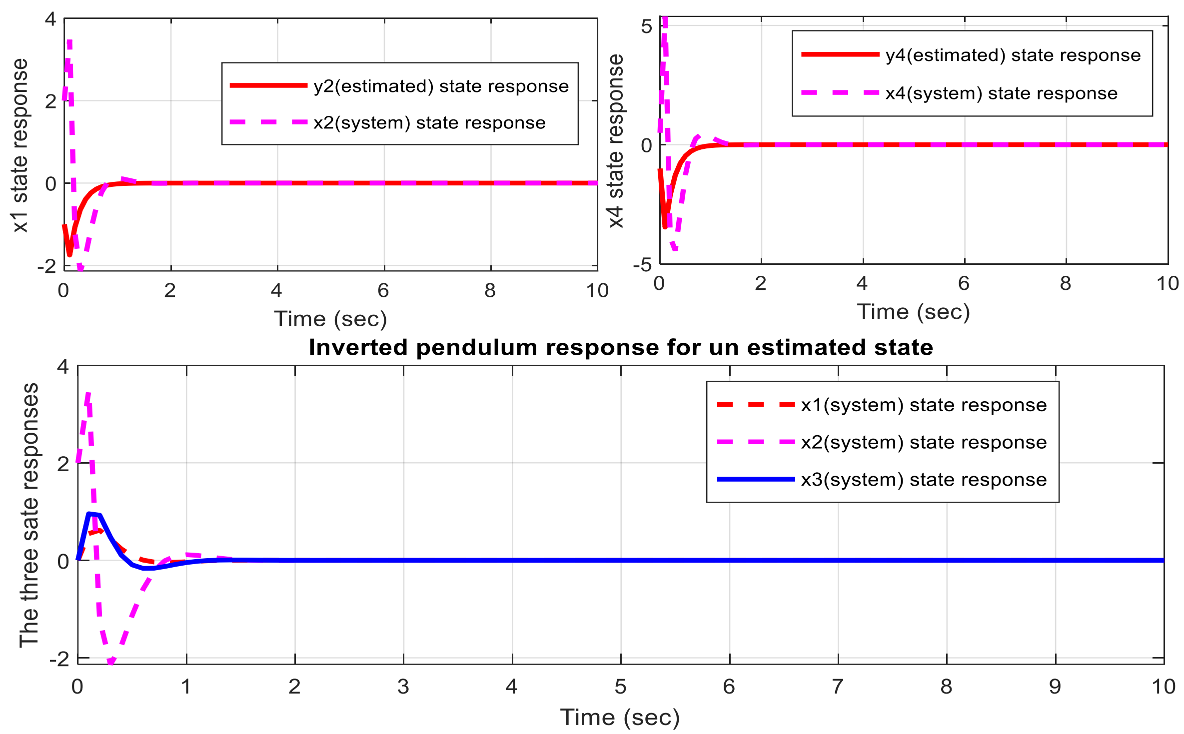

Figure 8. The response of the estimated and system response for initial conditions (0, 2, 0, 0.5)

It has been observed from the simulation result on *figure 7 and figure 8* that the angular velocity and cart speed tracks the system response for different initial condition by varying the desired pole location for the left-hand plane of the s-plane.

5. Conclusion

Modelling of an inverted pendulum shows that the system is unstable without a controller. Results of applying reduced order observer state feedback controllers using pole placement show that the system can be stabilized. The simulation of inverted pendulum based on reduced order pole placement design has been done on MATLAB/SIMULINK. It has been observed from the simulation result that the angular velocity and cart speed tracks the system response for different initial condition by varying the desired pole location for the left-hand plane of the s-plane. The estimation error for the angular velocity and cart speed of inverted pendulum have been observed on MATLAB SIMULINK for different initial condition. In general, if some of the system is unknown and the other state is known we can design using reduced order observer for any physical system.

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